Axial Dispersion with Time Variable Flow in Multiphase Systems

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A theory has been developed for heat or mass transfer between phases which may have unsteady time dependent velocity fields. The configuration analyzed involves two concentric tubes, but the analysis can be extended to more than two phases, or streams, and to noncylindrical cross sections. For the configuration considered, the dispersion coefficient is found from first principles.

It is noteworthy that the dispersion model given by Equation (15) or (63) and (64), which is consistent with the basic continuum model of Equations (8) and (9), is based on the superficial velocity rather than the interstitial velocity. This strongly suggests that experimental observations on packed beds, or other multiphase systems, should be interpreted on the basis of u_m, the superficial velocity, rather than the interstitial velocity which has been used often in the past.

Many important separation processes that are of interest to chemical engineers involve intimate contact between two or more phases all of which may be flowing as in gas-liquid absorption, or as in gas adsorption systems, one phase may be stationary. Such systems are usually studied by assuming steady state conditions exist, to simplify the analysis.

In recent years, both laminar and turbulent cocurrent and countercurrent steady state multistream heat exchangers have been analyzed (*2* to *5, 11* to *16*) mathematically. A review of the mathematical basis of this work and the relation of cocurrent analyses to earlier work on composite body heat transfer is given elsewhere (*5*). These studies show that steady state multistream systems can be analyzed by more or less classical methods but the analyses are complex and the numerical calculations necessary are rather tedious. The extension of these methods to apply to unsteady state systems promises to be even more complex. Therefore, one cannot expect to gain incisive physical insights from them without performing very extensive numerical calculations. Thus the development of a new, simpler method seems desirable.

It is the purpose of this work to:

1. Develop a method of analysis which enables one to deal with multiphase, time dependent flows, wherein the velocity distribution in each stream is described by continuous arbitrary functions given by $u_i(t, r)$.
2. Show clearly how the dispersion coefficient, and dispersion model, enter the problem of finding the solution for the local concentration distribution in each phase, or stream.
3. Show explicitly how the dispersion coefficient depends on the properties of the system.

The problem of axial dispersion in steady flow multiphase systems has been studied by Aris (*1*) who employed the method of moments to calculate a dispersion coefficient for such systems. In two other articles which are relevant, Gill (*7, 8*) analyzed unsteady convective diffusion in single stream steady and unsteady flow systems. The method developed by Gill for single phase systems will be further extended herein to unsteady flow multiphase systems.

ANALYSIS

The present analysis will consider convective diffusion of mass or energy in two concentric annular layers wherein the velocity fields are functions of transverse position and time. The same method can be extended to other configurations in a fairly straightforward way.

For the case of two dimensional systems, the governing differential equations are

$$\frac{\partial C_1}{\partial t} + u_1\,(t, r)\,\frac{\partial C_1}{\partial x} = \frac{\partial}{\partial x} D_{1x}(r)\,\frac{\partial C_1}{\partial x} + \frac{1}{r}\,\frac{\partial}{\partial r}\, r\, D_{1r}(r)\,\frac{\partial C_1}{\partial r} \tag{1}$$

$$\frac{\partial C_2}{\partial t} + u_2\,(t, r)\,\frac{\partial C_2}{\partial x} = \frac{\partial}{\partial x} D_{2x}(r)\,\frac{\partial C_2}{\partial x} + \frac{1}{r}\,\frac{\partial}{\partial r}\, r\, D_{2r}(r)\,\frac{\partial C_2}{\partial r} \tag{2}$$

and the boundary conditions are

$$\frac{\partial C_1}{\partial r}\,(t, x, 0) = \frac{\partial C_2}{\partial r}\,(t, x, R_2) = 0 \tag{3}$$

$$C_1\,(t, x, R_1) = H\; C_2\,(t, x, R_1) \tag{4}$$

$$D_1\,\frac{\partial C_1}{\partial r}\,(t, x, R_1) = D_2\,\frac{\partial C_2}{\partial r}\,(t, x, R_1) \tag{5}$$

If we use the transformation

$$x_1 = x - \int_0^t u_m(t)\;dt \tag{6}$$

where

$$u_m(t) = \frac{2}{R_2^2}\left[\int_0^{R_1} r u_1\,(t, r)\;dr + \int_{R_1}^{R_2} r u_2\,(t, r)\;dr\right] \tag{7}$$

then Equations (1) and (2) become

$$\frac{\partial C_1}{\partial t} + (u_1 - u_m)\,\frac{\partial C_1}{\partial x_1} = \frac{\partial}{\partial x_1} D_{1x}(r)\,\frac{\partial C_1}{\partial x_1} + \frac{1}{r}\,\frac{\partial}{\partial r}\, r\, D_{1r}(r)\,\frac{\partial C_1}{\partial r} \tag{8}$$

and

$$\frac{\partial C_2}{\partial t} + (u_2 - u_m)\,\frac{\partial C_2}{\partial x_1} = \frac{\partial}{\partial x_1} D_{2x}(r)\,\frac{\partial C_2}{\partial x_1} + \frac{1}{r}\,\frac{\partial}{\partial r}\, r\, D_{2r}(r)\,\frac{\partial C_2}{\partial r} \tag{9}$$

We now formulate the solutions of Equations (8) and (9) as

$$C_1 = HC_m + \sum_{k=1}^{\infty} f_k(t, r)\,\frac{\partial^k C_m}{\partial {x_1}^k} \tag{10}$$

and

$$C_2 = C_m + \sum_{k=1}^{\infty} g_k(t, r)\,\frac{\partial^k C_m}{\partial {x_1}^k} \tag{11}$$

where

$$C_m(t, x) = \frac{2}{R_2^2}\left[\frac{1}{H}\int_0^{R_1} r\ C_1 dr + \int_{R_1}^{R_2} r\ C_2 dr\right] \qquad (12)$$

which is a significant generalization of the solutions developed elsewhere (7, 8). If one substitutes Equations (10) and (11) in Equations (8) and (9) the result is

$$H\left[\frac{\partial C_m}{\partial t} + (u_1 - u_m)\frac{\partial C_m}{\partial x_1} - D_{1x}\frac{\partial^2 C_m}{\partial x_1^2}\right] + \sum_{k=1}^{\infty}\left\{\left[\frac{\partial f_k}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r} r\ D_{1r}\frac{\partial f_k}{\partial r}\right]\frac{\partial^k C_m}{\partial x_1^k} + (u_1 - u_m) f_k \frac{\partial^{k+1} C_m}{\partial x_1^{k+1}} - D_{1x} f_k \frac{\partial^{k+2} C_m}{\partial x_1^{k+2}} + f_k \frac{\partial^{k+1} C_m}{\partial t \partial x_1^k}\right\} = 0 \qquad (13)$$

and

$$\frac{\partial C_m}{\partial t} + (u_2 - u_m)\frac{\partial C_m}{\partial x_1} - D_{2x}\frac{\partial^2 C_m}{\partial x_1^2} + \sum_{k=1}^{\infty}\left\{\left[\frac{\partial g_k}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r} r\ D_{2r}\frac{\partial g_k}{\partial r}\right]\frac{\partial^k C_m}{\partial x_1^k} + (u_2 - u_m) g_k \frac{\partial^{k+1} C_m}{\partial x_1^{k+1}} - D_{2x}\ g_k \frac{\partial^{k+2} C_m}{\partial x_1^{k+2}} + g_k \frac{\partial^{k+1} C_m}{\partial t \partial x_1^k}\right\} = 0 \qquad (14)$$

Now, if the process of distributing C_m is assumed to be diffusive in nature for sufficiently large values of t, then one can write

$$\frac{\partial C_m}{\partial t} = K(t)\frac{\partial^2 C_m}{\partial x_1^2} \qquad (15)$$

and this requires

$$\frac{\partial^{k+1} C_m}{\partial t \partial x_1^k} = K\frac{\partial^{k+2} C_m}{\partial x_1^{k+2}} \qquad (16)$$

By substituting Equation (16) into Equations (13) and (14) and equating coefficients of $\partial^k C_m / \partial x_1^k$ to zero, one gets the following systems of equations:

$$\frac{\partial f_1}{\partial t} + H(u_1 - u_m) = \frac{1}{r}\frac{\partial}{\partial r} r\ D_{1r}\frac{\partial f_1}{\partial r} \qquad (17a)$$

$$\frac{\partial f_2}{\partial t} + (u_1 - u_m) f_1 + H(K - D_{1x}) = \frac{1}{r}\frac{\partial}{\partial r} r\ D_{1r}\frac{\partial f_2}{\partial r} \qquad (17b)$$

$$\frac{\partial f_{k+2}}{\partial t} + (u_1 - u_m) f_{k+1} + (K - D_{1x}) f_k = \frac{1}{r}\frac{\partial}{\partial r} r\ D_{1r}\frac{\partial f_{k+2}}{\partial r}, \quad k = 1, 2, 3 \ldots \qquad (17c)$$

and

$$\frac{\partial g_1}{\partial t} + (u_2 - u_m) = \frac{1}{r}\frac{\partial}{\partial r} r\ D_{2r}\frac{\partial g_1}{\partial r} \qquad (18a)$$

$$\frac{\partial g_2}{\partial t} + (u_2 - u_m) g_1 + (K - D_{2x}) = \frac{1}{r}\frac{\partial}{\partial r} r\ D_{2r}\frac{\partial g_2}{\partial r} \qquad (18b)$$

$$\frac{\partial g_{k+2}}{\partial t} + (u_2 - u_m) g_{k+1} + (K - D_{2x}) g_k = \frac{1}{r}\frac{\partial}{\partial r} r\ D_{2r}\frac{\partial g_{k+2}}{\partial r}, \quad k = 1, 2, 3 \ldots \qquad (18c)$$

The solution of Equations (15), (17), and (18) constitutes the solution of the problem of two phase unsteady convective diffusion with time variable flow. It will be shown that, as in the case of single phase systems, the dispersion coefficient, K, is a time dependent eigenvalue which we can determine explicitly. That is, K must assume the particular form given in Equation (62) or the solutions given by Equations (10) and (11) will not satisfy the boundary conditions, Equation (3) to (5), of the problem. The solution for first-order functions f_1 and g_1 essentially reduces to solving the transient composite body problem given by Equations (26) and (28) and the steady state composite body problem given by Equations (27) and (29). To obtain the higher order functions one would proceed along similar lines to those used to find f_1 and g_1.

First, one must find the solution of Equations (17*a*) and (18*a*) and this is most easily accomplished by using the Duhamel theorem (6). Let $F_1(\lambda, t, r)$ and $G_1(\lambda, t, r)$ be the solutions of

$$\frac{\partial F_1}{\partial t} + H[u_1(\lambda, r) - u_m(\lambda)] = \frac{1}{r}\frac{\partial}{\partial r} r\ D_{1r}\frac{\partial F_1}{\partial r} \qquad (19)$$

and

$$\frac{\partial G_1}{\partial t} + u_2(\lambda, r) - u_m(\lambda) = \frac{1}{r}\frac{\partial}{\partial r} r\ D_{2r}\frac{\partial G_1}{\partial r} \qquad (20)$$

with

$$F_1(\lambda, t, R_1) = HG_1(\lambda, t, R_1) \qquad (21)$$

$$D_1\frac{\partial F_1}{\partial r}(\lambda, t, R_1) = D_2\frac{\partial G_1}{\partial r}(\lambda, t, R_1) \qquad (22)$$

$$\frac{\partial F_1}{\partial r}(\lambda, t, 0) = \frac{\partial G_1}{\partial r}(\lambda, t, R_2) = 0 \qquad (23)$$

where λ is a parameter which is taken as a constant and replaces t in u_1, u_2 and u_m. One can find F_1 and G_1 by letting

$$F_1 = F_{1s}(\lambda, r) - F_{1t}(\lambda, t, r) \qquad (24)$$

$$G_1 = G_{1s}(\lambda, r) - G_{1t}(\lambda, t, r) \qquad (25)$$

such that

$$\frac{1}{r}\frac{\partial}{\partial r} r\ D_{1r}\frac{\partial F_{1t}}{\partial r} = \frac{\partial F_{1t}}{\partial t} \qquad (26)$$

$$\frac{1}{r}\frac{d}{dr} rD_{1r}\frac{dF_{1s}}{dr} = [u_1(\lambda, r) - u_m(\lambda)]\,H \qquad (27)$$

and

$$\frac{1}{r}\frac{\partial}{\partial r} rD_{2r}\frac{\partial G_{1t}}{\partial r} = \frac{\partial G_{1t}}{\partial t} \qquad (28)$$

$$\frac{1}{r}\frac{d}{dr} rD_{2r}\frac{dG_{1s}}{dr} = u_2(\lambda, r) - u_m(\lambda) \qquad (29)$$

It follows immediately that

$$F_{1s} = F_{1s}(\lambda, 0) + H\int_0^r \frac{1}{\xi D_{1r}}\int_0^\xi \sigma[u_1(\lambda, \sigma) - u_m(\lambda)]\,d\sigma\,d\xi \qquad (30)$$

and

$$G_{1s} = G_{1s}(\lambda, R_2) + \int_{R_2}^r \frac{1}{\xi D_{2r}}\int_{R_2}^\xi \sigma[u_2(\lambda, \sigma) - u_m(\lambda)]\,d\sigma\,d\xi \qquad (31)$$

The conditions of Equation (21) and

$$\int_0^{R_1} rF_1(\lambda, t, r)\,dr + \int_{R_1}^{R_2} rG_1(\lambda, t, r)\,dr = 0 \qquad (32)$$

which follows from Equations (10), (11), and (12), require that

$$F_{1s}(\lambda, 0) = -\frac{HA + \dfrac{(R_2^2 - R_1^2)}{2} B}{(1-H)\dfrac{R_1^2}{2} - \dfrac{R_2^2}{2}} \qquad (33)$$

$$G_{1s}(\lambda, R_2) = \frac{\dfrac{R_1^2}{2} B - A}{(1-H)\dfrac{R_1^2}{2} - \dfrac{R_2^2}{2}} \qquad (34)$$

where

$$A = -\left\{ H\int_0^{R_1} r \int_0^r \frac{1}{\xi D_{1r}} \int_0^\xi \sigma[u_1(\lambda, \sigma) - u_m(\lambda)]\, d\sigma\; d\xi\; dr + \int_{R_1}^{R_2} r \int_{R_2}^r \frac{1}{\xi D_{2r}} \int_{R_2}^\xi \sigma[u_2(\lambda, \sigma) - u_m(\lambda)]\, d\sigma\; d\xi\; dr \right\} \qquad (35)$$

and

$$B = H\left\{ -\int_0^{R_1} \frac{1}{\xi D_{1r}} \int_0^\xi r[u_1(\lambda, \sigma) - u_m(\lambda)]\, d\sigma\; d\xi + \int_{R_2}^{R_1} \frac{1}{\xi D_{2r}} \int_{R_2}^\xi \sigma[u_2(\lambda, \sigma) - u_m(\lambda)]\, d\sigma\; d\xi \right\} \qquad (36)$$

For either laminar or turbulent flows in which D_{1r} and D_{2r} are functions of r, to find solutions for F_{1t} and G_{1t} we will again use the Duhamel theorem and thus their values at R_1, $F_{1t}\ (\lambda, t, R_1)$ and $G_{1t}\ (\lambda, t, R_1)$, are taken to be $F_{1t}(\lambda, \phi, R_1)$ and $G_{1t}(\lambda, \phi, R_1)$ which are considered constants since ϕ is taken to be a constant parameter. One can solve Equations (26) and (28) by using the Duhamel theorem and we obtain

$$F_{1t} = \frac{\partial}{\partial t} \int_0^t [\theta_1(\phi, t - \phi, r) + F_{1t}(\phi, R_1)]\, d\phi \qquad (37)$$

$$G_{1t} = \frac{\partial}{\partial t} \int_0^t [\theta_2(\phi, t - \phi, r) + G_{1t}(\phi, R_1)]\, d\phi \qquad (38)$$

Equations (37) and (38) apply for either laminar or turbulent flow. For turbulent flow the eigenfunctions and eigenvalues must be determined numerically whereas for laminar flow the solutions are given in terms of Bessel functions. Therefore, to demonstrate the method we consider the laminar flow case and obtain for θ_1 and θ_2 the following results

$$\theta_1 = \sum_{n=1}^{\infty} [a_{1n} - a_{2n}\ F_{1t}(\lambda, \phi, R_1)]\, e^{-\gamma_n^2 t} J_0\left(\frac{\gamma_n r}{\sqrt{D_1}}\right) \qquad (39)$$

and

$$\theta_2 = \sum_{n=1}^{\infty} [b_{1n} - b_{2n}\ G_{1t}(\lambda, \phi, R_1)]\, e^{-\alpha_n^2 t} \psi_n\left(\frac{\alpha_n r}{\sqrt{D_2}}\right) \qquad (40)$$

The eigenfunction $J_0(\gamma_n r/\sqrt{D_1})$ is a zero order Bessel function of the first kind and ψ_n is defined by

$$\psi_n\left(\frac{\alpha_n r}{\sqrt{D_2}}\right) = J_0\left(\frac{\alpha_n r}{\sqrt{D_2}}\right) - \frac{J_0\left(\dfrac{\alpha_n R_1}{\sqrt{D_2}}\right)}{Y_0\left(\dfrac{\alpha_n R_1}{\sqrt{D_2}}\right)} Y_0\left(\frac{\alpha_n r}{\sqrt{D_2}}\right) \qquad (41)$$

with $Y_0(\alpha_n r/\sqrt{D_2})$ being a zero order Bessel function of the second kind. The eigenvalues γ_n and α_n are the zeros of

$$J_0\left(\frac{\gamma_n}{\sqrt{D_1}} R_1\right) = 0 \qquad (42)$$

and

$$Y_0\left(\frac{\alpha_n R_1}{\sqrt{D_2}}\right) J_1\left(\frac{\alpha_n R_2}{\sqrt{D_2}}\right) = J_0\left(\frac{\alpha_n R_1}{\sqrt{D_2}}\right) Y_1\left(\frac{\alpha_n R_2}{\sqrt{D_2}}\right) \qquad (43)$$

Furthermore, the coefficients are given by

$$a_{1n} = \frac{\int_0^{R_1} rF_{1s}(\lambda, r) J_0\, dr}{\int_0^{R_1} r\, J_0^2\, dr}, \quad a_{2n} = \frac{\int_0^{R_1} rJ_0 dr}{\int_0^{R_1} rJ_0^2 dr} \qquad (44)$$

$$b_{1n} = \frac{\int_{R_1}^{R_1} rG_{1s}(\lambda, r)\, \psi_n dr}{\int_{R_1}^{R_2} r\, \psi_n^2\, dr}, \quad b_{2n} = \frac{\int_{R_1}^{R_2} r\psi_n dr}{\int_{R_1}^{R_2} r\psi_n^2 dr} \qquad (45)$$

Now it remains to find $F_{1t}(t, R_1)$ and $G_{1t}(t, R_1)$. First, we use Equation (21) and then, because of Equation (22), equate the derivatives of Equations (37) and (38) multiplied by D_1 and D_2 respectively. This yields

$$D_1 \frac{\partial}{\partial t} \int_0^t \left\{ \sum_{n=1}^{\infty} [a_{1n} - a_{2n}\, HG_{1t}(\lambda, \phi, R_1)]\, e^{-\gamma_n^2(t-\phi)} \frac{dJ_0}{dr}(R_1) \right\} d\phi = D_2 \frac{\partial}{\partial t} \int_0^t \left\{ \sum_{n=1}^{\infty} [b_{1n} - b_{2n}\, G_{1t}(\lambda, \phi, R_1)]\, e^{-\alpha_n^2(t-\phi)} \frac{d\psi_n}{dr}(R_1) \right\} d\phi \qquad (46)$$

Equation (46) is a Volterra integral equation and is most easily solved by using the LaPlace transform. If one notes that the integrals are convolution integrals and takes the transform of Equation (46) the result is

$$\overline{G_{1t}}(p, R_1) = \frac{\displaystyle\sum_{n=1}^{\infty} \left[\frac{a_{1n}D_1 \dfrac{dJ_0}{dr}(R_1)}{p + \gamma_n^2} - \frac{b_{1n}D_2 \dfrac{d\psi_n}{dr}(R_1)}{p + \alpha_n^2} \right]}{p \displaystyle\sum_{n=1}^{\infty} \left[\frac{a_{2n}HD_1 \dfrac{dJ_0}{dr}(R_1)}{p + \gamma_n^2} - \frac{b_{2n}D_2 \dfrac{d\psi_n}{dr}(R_1)}{p + \alpha_n^2} \right]} \qquad (47)$$

The inverse transform of $\overline{G}_{1t}(p, R_1)$ is found easily by the method of residues to be

$$G_{1t}(t, R_1) = \sum_{m=0}^{\infty} \frac{\displaystyle\sum_{n=1}^{\infty} \left[\frac{a_{1n}D_1 \dfrac{dJ_0}{dr}(R_1)}{\gamma_n^2 - \beta_m^2} - \frac{b_{1n}D_2 \dfrac{d\psi_n}{dr}(R_1)}{\alpha_n^2 - \beta_m^2} \right] e^{-\beta_m^2 t}}{\displaystyle\sum_{n=1}^{\infty} \left[\frac{a_{2n}H\gamma_n^2 D_1 \dfrac{dJ_0}{dr}(R_1)}{(\gamma_n^2 - \beta_m^2)^2} - \frac{b_{2n}\alpha_n^2 D_2 \dfrac{d\psi_n}{dr}(R_1)}{(\alpha_n^2 - \beta_m^2)^2} \right]} \qquad (48)$$

where the $(-\beta_m^2)$ are the poles of $\overline{G}_{1t}(p, R_1)$ which can be determined from

$$\sum_{n=1}^{\infty} \beta_m^2 \left[\frac{a_{2n} H D_1 \dfrac{dJ_0}{dr}(R_1)}{\gamma_n^2 - \beta_m^2} - \frac{b_{2n} D_2 \dfrac{d\psi_n}{dr}(R_1)}{\alpha_n^2 - \beta_m^2} \right] = 0 \tag{49}$$

so it is clear $\beta_0 = 0$ and the other poles can be found by using the Newton-Raphson method. It is worth noting that since the smallest pole is zero the values of the functions at the interface, R_1, are constant for larger values of time. With F_{1t} and G_{1t} now known, we can calculate f_1 and g_1 by using

$$f_1 = \frac{\partial}{\partial t} \int_0^t F_1(\lambda, t-\lambda, r) d\lambda \tag{50}$$

$$g_1 = \frac{\partial}{\partial t} \int_0^t G_1(\lambda, t-\lambda, r) d\lambda \tag{51}$$

With f_1 and g_1 known for systems with arbitrary velocity distributions and total diffusion coefficients, we can now proceed to consider the next functions, f_2 and g_2, which involve the dispersion coefficient K. As in the single stream problem, the determination of K is of crucial importance since this enables one to find C_m which together with f_1, g_1 and $\partial C_m/\partial x_1$ provides the means to obtain the first approximation for the local distributions in each phase, C_1 and C_2.

To solve Equations (17*b*) and (18*b*), let F_2 and G_2 satisfy

$$\frac{\partial F_2}{\partial t} + [u_1(\lambda, r) - u_m(\lambda)] f_1(\lambda, r) + H[K(\lambda) - D_{1x}] = \frac{1}{r} \frac{\partial}{\partial r} r D_{1r} \frac{\partial F_2}{\partial r} \tag{52}$$

and

$$\frac{\partial G_2}{\partial t} + [u_2(\lambda, r) - u_m(\lambda)] g_2(\lambda, r) + K(\lambda) - D_{2x} = \frac{1}{r} \frac{\partial}{\partial r} r D_{2r} \frac{\partial G_2}{\partial r} \tag{53}$$

with the boundary conditions given by Equations (21), (22), and (23) with the subscripts changed from 1 to 2. Again we separate the functions such that

$$F_2 = F_{2s}(\lambda, r) - F_{2t}(\lambda, t, r) \tag{54}$$

and

$$G_2 = G_{2s}(\lambda, r) - G_{2t}(\lambda, t, r) \tag{55}$$

where

$$\frac{1}{r} \frac{d}{dF} r D_{1r} \frac{dF_{2s}}{dr} = [u_1(\lambda, r) - u_m(\lambda)] f_1(\lambda, r) + H[K(\lambda) - D_{1x}] \tag{56}$$

$$\frac{1}{r} \frac{\partial}{\partial r} r D_{1r} \frac{\partial F_{2t}}{\partial r} = \frac{\partial F_{2t}}{\partial t} \tag{57}$$

and

$$\frac{1}{r} \frac{d}{dr} r D_{2r} \frac{dG_{2s}}{dr} = [u_2(\lambda, r) - u_m(\lambda)] g_1(\lambda, r) + K(\lambda) - D_{2x} \tag{58}$$

$$\frac{1}{r} \frac{\partial}{\partial r} r D_{2r} \frac{\partial G_{2t}}{\partial r} = \frac{\partial G_{2t}}{\partial t} \tag{59}$$

An explicit expression for the dispersion coefficient, K, can be deduced from Equations (56) and (58). One integration of these equations yields

$$D_{1r} \frac{dF_{2s}}{dr} = \frac{1}{R_1} \int_0^{R_1} r\{[u_1(\lambda, r) - u_m(\lambda)] f_1(\lambda, r) + H[K(\lambda) - D_{1x}]\} dr \tag{60}$$

and

$$D_{2r} \frac{dG_{2s}}{dr} = \frac{1}{R_1} \int_{R_2}^{R_1} r\{[u_2(\lambda, r) - u_m(\lambda)] g_1(\lambda, r) + K(\lambda) - D_{2x}\} dr \tag{61}$$

On the basis of Equation (22) one can equate the right-hand sides of Equations (60) and (61) and thereby obtain the expression for $K(t)$ as

$$K(t) = \frac{2}{R_2^2 - (1-H) R_1^2} \left\{ \left[\int_0^{R_1} Hr\, D_{1x} dr + \int_{R_1}^{R_2} r\, D_{2x} dr \right] - \left[\int_0^{R_1} r[u_1(t, r) - u_m(t)] f_1(t, r)\, dr + \int_{R_1}^{R_2} r[u_2(t, r) - u_m(t)] g_1(t, r)\, dr \right] \right\} \tag{62}$$

Note that it is not necessary to determine f_2 and g_2 in order to evaluate K. However it is straightforward to calculate f_2 and g_2 as well as higher order f_k and g_k if one wishes to improve the approximation of the local concentrations (or temperatures), C_1 and C_2, but a considerable amount of information of practical value can be obtained without doing this.

For the special case of steady flow in Turner structures, which may be viewed as a two region problem, wherein the physical properties are the same in both regions, Equation (62) reduces to equation (28) of reference 9. In (9) it was shown by comparing the analytical results with finite difference calculations that for Turner structures the local distributions are predicted well by the first two terms of Equations (10) and (11) if dimensionless time is sufficiently large. A similar conclusion was reached elsewhere (*10*) for the dispersion of slugs in tubes.

PHYSICAL INTERPRETATION OF THE ANALYSIS

The next important point to be considered is how to apply the preceding analysis to systems of practical interest. Aris (*1*) has discussed some steady flow applications of the method of moments analysis to distillation and chromatography and, of course, the present analysis is equally applicable to these cases. The particular application chosen will determine the initial condition at $t = 0$ and the x boundary conditions to be chosen for Equations (1) and (2). One interesting aspect of the dispersion theory is that the $t = 0$ and x boundary conditions do not influence K.

Other applications of the present analysis are rather numerous. If the boundary conditions are chosen properly both cocurrent and countercurrent systems can be considered. In such cases it is useful to note that Equations (12) and (15) imply

$$\frac{\partial C_{m1}}{\partial t} + u_m \frac{\partial C_{m1}}{\partial x} = K(t) \frac{\partial^2 C_{m1}}{\partial x^2} \tag{63}$$

$$\frac{\partial C_{m2}}{\partial t} + u_m \frac{\partial C_{m2}}{\partial x} = K(t) \frac{\partial^2 C_{m2}}{\partial x^2} \tag{64}$$

where

$$C_{m1} = \frac{2}{R_1^2} \int_0^{R_1} r\, C_1 dr, \quad C_{m2} = \frac{2}{R_2^2 - R_1^2} \int_{R_1}^{R_2} r\, C_2 dr$$

and

$$C_m = \frac{1}{H} - \frac{R_1^2}{R_2^2} C_{m1} + \left(1 - \frac{R_1^2}{R_2^2}\right) C_{m2} \qquad (65)$$

Equations (63) and (64) also apply when one phase is stationary as it is in several applications of interest in chemical engineering. Thus the present analysis also can be applied to either heat transfer or gas adsorption in packed beds but it is very difficult to calculate the local velocity field because of the complex configuration involved. However one method of modeling such systems is in terms of tube bundles and the application to a single tube is straightforward.

For example, consider unsteady heat transfer to the wall of an insulated pipe so that there is no heat flux through the outer wall surface. This may be of interest *per se*, or as a model of a packed bed wherein the tube wall models the packing and the open cross section models the interstices. The K can be calculated from Equation (62) with R_1 the inner wall radius and R_2 the outer wall radius. Suppose, for definiteness, that a slug of fluid of length x_s and temperature T_1, originally placed with its center at $x = 0$, is started from rest and passes through a pipe which originally had a wall temperature T_2 and the fluid surrounding the slug is at temperature T_0. In this case C_m is given by

$$C_m = \frac{R_1^2}{R_2^2} T_0 + \left(1 - \frac{R_1^2}{R_2^2}\right) T_2 + \left[\frac{R_1^2}{R_2^2}\,\frac{(T_1 - T_0)}{2} + \left(1 - \frac{R_1^2}{R_2^2}\right)\left(\frac{T_1 - T_2}{2}\right)\right] \left\{ \operatorname{erf} \frac{\frac{x_s}{2} + x_1}{2\sqrt{\int_0^t K dt}} + \operatorname{erf} \frac{\frac{x_s}{2} - x_1}{2\sqrt{\int_0^t K dt}} \right\} \qquad (66)$$

With the function C_m known, one can calculate the local distributions C_1 and C_2 and any other information regarding the transfer process between phases.

CONCLUSIONS

1. For sufficiently large values of time a dispersion theory has been developed for multi-phase systems with time variable flow and the dispersion coefficient was found from first principles to be given by Equation (62).
2. The theory enables one to calculate the local as well as the average concentration or temperature distributions, and therefore interphase transfer rates, for both cocurrent and countercurrent flows. Clearly, a stationary phase such as an adsorption bed is a special case of these more general systems.
3. The theory can be extended to more than two phases, or streams, and to noncylindrical cross sections.
4. It is particularly noteworthy that the dispersion model, given by Equation (15) or (63) and (64), which is consistent with the basic continuum model of Equations (8) and (9), is based on the superficial velocity rather than the interstitial velocity. This strongly suggests that experimental observations should be interpreted on the basis of u_m rather than the interstitial velocity which has been used often in the past.

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NOTATION

a_{1n}, a_{2n} = coefficients defined by Equation (44)
b_{1n}, b_{2n} = coefficients defined by Equation (45)
C_1, C_2 = local concentration of phases 1 and 2
C_m = defined by Equation (12)
D_{1r}, D_{2r} = total radial diffusion coefficient of phases 1 and 2
D_{1x}, D_{2x} = total axial diffusion coefficient of phases 1 and 2
f_k = functions defined by Equation (10)
F_1 = function defined by Equations (19) and (50)
F_{1s} = function defined by Equations (24) and (29)
F_{1t} = function defined by Equations (24) and (26)
F_2 = defined by Equation (52)
F_{2s} = defined by Equations (54) and (56)
F_{2t} = defined by Equations (54) and (57)
g_k = functions defined by Equation (11)
G_{1s} = function defined by Equations (25) and (29)
G_{1t} = function defined by Equations (25) and (28)
G_2 = defined by Equation (53)
G_{2s} = defined by Equations (55) and (58)
G_{2t} = defined by Equations (55) and (59)
H = distribution constant defined by Equation (4)
J_0 = zero-order Bessel function of the first kind
K = dispersion coefficient defined by Equation (15)
r = radial distance
R_1 = thickness of phase 1
R_2 = thickness of phase 1 plus phase 2
t = time
u_1, u_2 = local velocity distributions of phases 1 and 2
u_m = defined by Equation (7)
x = axial distance
x_1 = defined by Equation (6)
Y_0 = zero-order Bessel function of the second kind

Greek Letters

α_n = eigenvalues defined by Equation (43)
β_m = poles of $\overline{G}_{1t}(p, R_1)$ given by Equation (49)
γ_n = eigenvalues defined by Equation (42)
θ_1, θ_2 = function defined by Equations (37), (38), (39), and (40)
λ = dummy variable
ϕ = dummy variable
ψ_n = defined by Equation (41)

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